Definition

Given a function $f : [-\pi, \pi] \to \mathbb{R}$.

- The Fourier coefficients are given by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) e^{-iny}\, dy$$

- The Fourier Series of f is given by

$$f \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$$

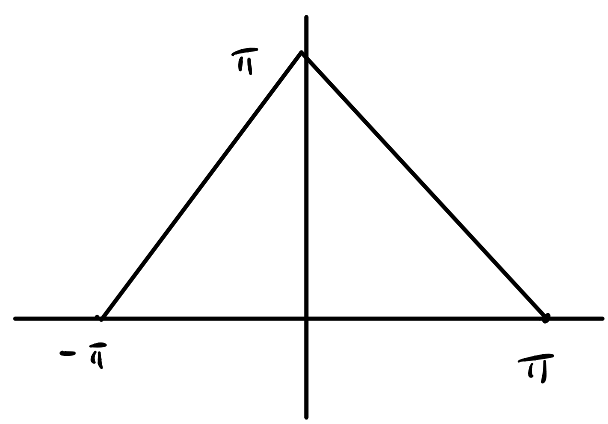
- The N-th partial sum of the Fourier Series of f is given by

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$$

$\boxed{\text{Compute Fourier Series and judge the convergence property}}$

(I) Define $f : [-\pi, \pi] \to \mathbb{R}$ by

$$f(x) = \pi - |x|$$



For $n \neq 0$, $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) e^{iny}\, dy$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(y)\cos ny\, dy + i\int_{-\pi}^{\pi} f(y)\sin ny\, dy\right.$$

(since f is even, $\cos ny$ is even, $\sin ny$ is odd.)

$$= \frac{1}{\pi}\int_0^{\pi} f(y)\cos ny \, dy$$

$$= \frac{1}{\pi}\int_0^{\pi} (\pi - y)\cos ny \, dy$$

$$= \int_0^{\pi} \cos ny \, dy - \int_0^{\pi} \frac{y}{\pi}\cos ny \, dy$$

integral by part

$$= \frac{\sin ny}{n}\Big|_0^{\pi} - \left(\frac{y \sin ny}{n\pi}\Big|_0^{\pi} - \int_0^{\pi} \frac{\sin ny}{n\pi} \, dy \right.$$

$$= \int_0^{\pi} \frac{\sin ny}{n\pi} \, dy$$

$$= -\frac{\cos ny}{n^2\pi}\Big|_0^{\pi}$$

$$= \frac{1-(-1)^n}{n^2\pi}$$

For $n \neq 0$, $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) \, dy$

$$= \frac{1}{\pi}\int_0^{\pi} (\pi - x) \, dy$$

$$= \frac{1}{\pi}\left(\pi x - \frac{x^2}{2}\right)\Big|_0^{\pi}$$

$$= \frac{\pi}{2}$$

Thus, $S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$

$$= \frac{\pi}{2} + \sum_{n=0}^{N} \frac{1-(-1)^n}{\pi n^2} \left(e^{inx} + e^{-inx}\right)$$

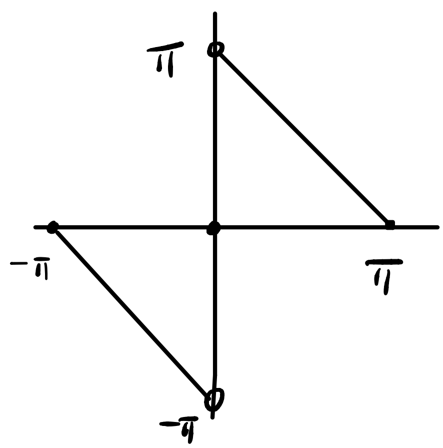
Therefore, $\left| S_N(f)(x) \right| \leq \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{N} \frac{1}{n^2} < \infty$ as $N \to \infty$.

Hence, the Fourier Series of f is uniformly convergent.

$\square$

(II) Define $f : [-\pi, \pi] \to \mathbb{R}$ by

$$f(x) = \begin{cases} -\pi - x, & \text{if } -\pi \leq x < 0, \\ 0, & \text{if } x = 0, \\ \pi - x, & \text{if } 0 < x < \pi. \end{cases}$$



For $n \neq 0$, $\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{iny} \, dy$

$$= \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} f(y) \cos ny \, dy + i \int_{-\pi}^{\pi} f(y) \sin ny \, dy \right)$$

(Since f is odd, $\cos ny$ is even, $\sin ny$ is odd.)

$$= \frac{i}{\pi} \int_0^{\pi} f(y) \sin ny \, dy$$

$$= \frac{i}{\pi} \int_0^{\pi} (\pi - y) \sin ny \, dy$$

$$= i\int_0^{\pi} \sin ny \, dy - \int_0^{\pi} \frac{iy\sin ny}{\pi} dy$$

Integral by part

$$= \frac{-i\cos ny}{n}\Big|_0^{\pi} - \left(\frac{-iy\cos ny}{n\pi}\Big|_0^{\pi} - \int_0^{\pi} \frac{-i\cos ny}{n\pi} dy \right)$$

$$= \frac{i(1-(-1)^n)}{n} + \frac{i\pi(-1)^n}{n\pi} + 0$$

$$= \frac{i}{n}$$

For $n=0$, $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)dy = 0$ since f is odd.

Thus $S_N(f)(x) = \sum_{n=1}^{N}\left(\frac{i}{n}e^{inx} - \frac{i}{n}e^{-inx}\right)$

$$= \sum_{n=1}^{N} \frac{i}{n}(2i\sin nx)$$

$$= -2\sum_{n=1}^{N} \frac{\sin nx}{n}$$

Dirichlet's Test.

If (a_n) is a sequence of real numbers and (b_n) is a sequence of complex numbers satisfying

- $a_n \downarrow 0$ as $n \to \infty$
- $\left|\sum_{n=1}^{N} b_n\right| \leq M$ for any $N \in \mathbb{N}$,

then $\sum_{n=1}^{\infty} a_n b_n < \infty$

For any $x \neq 0$, let $a_n = \frac{1}{n}$ and $b_n = e^{inx}$.

Clearly, $a_n \downarrow 0$ as $n \to \infty$.

$$\left| \sum_{n=1}^{N} b_n \right| = \left| \sum_{n=1}^{N} e^{inx} \right| = \left| \frac{1-e^{iNx}}{1-e^{ix}} \right| < \frac{2}{|1-e^{ix}|} < \infty$$

By Dirichlet's Test, $\sum_{n=1}^{\infty} \frac{i}{n} e^{inx} < \infty$.

Similarly, $\sum_{n=1}^{\infty} \frac{-i}{n} e^{-inx} < \infty$.

Therefore, the Fourier series of f converges for any $x \neq 0$.

Note that $S_N(f)(0) = 0$ for any N.

Hence, the Fourier series of f converges for any $x \in [-\pi, \pi]$.

$\square$

Definition

Given a series $\sum_{n=1}^{\infty} c_n$ of complex numbers.

Let $s_n := \sum_{k=1}^{n} c_k$, $\sigma_N := \frac{1}{N}\sum_{n=1}^{N} s_n$ and $A(r) = \sum_{n=1}^{\infty} c_n r^n$.

- $\sum_{n=1}^{\infty} c_n$ is Cesàro summable to s if $\sigma_N \to s$ as $N \to \infty$
- $\sum_{n=1}^{\infty} c_n$ is Abel summable to s if $A(r)$ converges for any $r \in [0,1)$ and $\lim_{r \to 1^-} A(r) = s$.

(I) Show that

Convergence $\Rightarrow$ Cesàro Summability $\Rightarrow$ Abel Summability

and none of the implications can be reserved.

(i) Convergence $\Rightarrow$ Cesàro Summability

$$s_n \to s \text{ as } n \to \infty \Rightarrow \frac{1}{N}\sum_{n=1}^{N} s_n \to s \text{ as } n \to \infty$$

(ii) Cersàro Summability $\not\Rightarrow$ Convergence

If $s_n = (-1)^n$, then s_n diverges as $n \to \infty$

and $\sigma_N = \frac{1}{N}\sum_{n=1}^{N} s_n \to 0$ as $N \to \infty$.

To construct such a sequence, let

$$(-1)^n = s_n = \sum_{k=1}^{n} c_k \text{ for any } n.$$

Then $c_1 = s_1 = -1$ and $c_n = s_n - s_{n-1} = (-1)^n - (-1)^{n-1}$

$$= (-1)^n \cdot 2 \quad \text{for } n \geq 2.$$

(iii) Cesàro Summability $\Rightarrow$ Abel Summability

Note that $\sum_{n=1}^{N} c_n r^n$

since $c_n = s_n - s_{n-1}$ $\quad = \sum_{n=1}^{N} (s_n - s_{n-1}) r^n$

$$= \sum_{n=1}^{N} s_n r^n - \sum_{n=0}^{N-1} s_n r^{n+1}$$

$$= \sum_{n=1}^{N-1} s_n (r^n - r^{n+1}) + s_N r^N$$

$$= (1-r) \sum_{n=1}^{N-1} s_n r^n + s_N r^N$$

since $s_n = n\sigma_n - (n-1)_n$ $\quad = (1-r) \sum_{n=1}^{N-1} [n\sigma_n - (n-1)\sigma_{n-1}] r^n + s_N r^N$

$$= (1-r) \left(\sum_{n=1}^{N-1} n\sigma_n r^n - \sum_{n=1}^{N-2} n\sigma_n r^{n+1} \right) + s_N r^N$$

$$= (1-r)^2 \sum_{n=1}^{N-2} n\sigma_n r^n + (N-1)\sigma_N r^{n-1} + s_N r^N$$

Since $\lim_{n\to\infty} n r^n = 0$ for $r \in (0,1)$, $A(r) = (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n$

Since $\sigma_n \to s$ as $n \to \infty$, to show $A(r) < \infty$, it suffices

to show $\sum_{n=1}^{\infty} n r^n < \infty$

Notice that $\sum_{n=1}^{N} n r^n = \sum_{n=1}^{N} \sum_{k=n}^{N} r^n$

$$= \sum_{n=1}^{N} r^n \left(\frac{1 - r^{N-n}}{1-r} \right)$$

$$= \frac{1}{1-r} \sum_{n=1}^{N} (r^n - r^N)$$

$$= \frac{r}{(1-r)^2} - \frac{r^N}{(1-r)^2} - \frac{N r^N}{1-r} \to \frac{r}{(1-r)^2} \text{ as } N \to \infty.$$

Thus $A(r) = (1-r)^2 \sum_{n=1}^{\infty} n \sigma_n r^n$ converges for any $r \in (0, 1)$.

$$\left| A(r) - s \right| \leq \left| A(r) - (1-r)^2 \sum_{n=1}^{\infty} n s r^n \right| + \left| (1-r)^2 \sum_{n=1}^{\infty} n s r^n - s \right|$$

$$= \left| (1-r)^2 \sum_{n=1}^{\infty} n(\sigma_n - s) r^n \right| + \left| rs - s \right|$$

Take $N \in \mathbb{N}$ s.t. $|\sigma_n - s| < \varepsilon$ for $n \geq N$

$$\leq \left| (1-r)^2 \sum_{n=1}^{N} n(\sigma_n - s) r^n \right| + \left| (1-r)^2 \sum_{n=N+1}^{\infty} n(\sigma_n - s) r^n \right| + s|r-1|$$

Take $r_0 < 1$ s.t. $(1-r_0)^2 < \frac{\varepsilon}{\sum_{n=1}^{N} n(\sigma_n - s)}$ and $1 - r_0 < \frac{1}{s}$

$$\leq \varepsilon + r\varepsilon + \varepsilon \leq 3\varepsilon \text{ for any } r \in (r_0, 1).$$

Hence, $A(r) \to s$ as $r \to 1^-$, i.e.,

$\sum_{n=1}^{\infty} c_n$ is Abel summable to s.

(iv) Abel summability $\not\Rightarrow$ Cesàro summability

If $\sigma_n = \begin{cases} 0, & n \text{ is odd}, \\ 1, & n \text{ is even}, \end{cases}$ then

$$A(r) = (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n$$

$$= (1-r)^2 \sum_{n=1}^{\infty} 2n\, r^{2n}$$

$$= 2(1-r)^2 \sum_{n=1}^{\infty} (r^2)^n$$

$$= 2(1-r)^2 \frac{r^2}{(1-r^2)^2}$$

$$= 2\frac{r^2}{(1+r)^2} \to \frac{1}{2} \quad \text{as } r \to 1^-$$

Thus $\sum_{n=1}^{\infty} C_n$ is Abel summable to 1 but not Cesàro summable.

To construct such a sequence, recall

$$S_n = n\sigma_n - (n-1)\sigma_{n-1} = \begin{cases} n, & n \text{ is even} \\ -(n-1), & n \text{ is odd} \end{cases} \quad \text{and}$$

$$C_n = S_n - S_{n-1} = \begin{cases} 2n-2, & n \text{ is even} \\ -(2n-2), & n \text{ is odd} \end{cases}$$

$$\{0, 2, -4, 6, -8, \dots\}$$

$\square$

(II) Suppose $nc_n \to 0$ as $n \to \infty$. Then

Convergence $\Longleftrightarrow$ Cesàro Summability $\Longleftrightarrow$ Abel Summability

Tauberian Theorem

If $nc_n \to 0$, then Abel summability implies convergence.

Proof:

$$\left| \sum_{n=1}^{N} c_n - A(r) \right|$$

$$\leq \left| \sum_{n=1}^{N} c_n - \sum_{n=1}^{N} c_n r^n \right| + \left| \sum_{n=N+1}^{\infty} c_n r^n \right|$$

$$= \left| \sum_{n=1}^{N} c_n (1-r^n) \right| + \left| \sum_{n=N+1}^{\infty} c_n r^n \right|$$

$$= \left| (1-r) \sum_{n=1}^{N} |c_n| n \right| + \left| \sum_{n=N+1}^{\infty} c_n r^n \right|$$

$$\leq \frac{1}{N} \sum_{n=1}^{N} |c_n| n + N \sup_{n>N} |c_n|$$ if $r = 1 - \frac{1}{N}$

Therefore $\left| \sum_{n=1}^{N} c_n - s \right| \leq \left| \sum_{n=1}^{N} c_n - A(1-\frac{1}{N}) \right| + \left| A(1-\frac{1}{N}) - s \right|$

since $c_n n \to 0$, $\frac{1}{N} \sum_{n=1}^{N} |c_n| n \to 0$

$$\leq \left| \frac{1}{N} \sum_{n=1}^{N} |c_n| n \right| + N \sup_{n>N} |c_n| + \left| A(1-\tfrac{1}{N}) - s \right|$$

Take N s.t. $\frac{1}{N} \sum_{n=1}^{N} |c_n| n < \varepsilon$, $|A(1-\frac{1}{N}) - s| < \varepsilon$ and $n|c_n| < \varepsilon$ for any $n > N$

$$\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

Hence, $\sum_{n=1}^{\infty} c_n = S$

$\square$